

Continuous Markovian Logic - From Complete Axiomatization to the Metric Space of Formulas*

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Abstract

Continuous Markovian Logic (CML) is a multimodal logic that expresses quantitative and qualitative properties of continuous-space and continuous-time labelled Markov processes (CMPs). The modalities of CML approximate the rates of the exponentially distributed random variables that characterize the duration of the labeled transitions. In this paper we present a sound and complete Hilbert-style axiomatization of CML for the CMP-semantics and prove some meta-properties including the small model property. CML characterizes stochastic bisimulation and supports the definition of a quantified extension of satisfiability relation that measures the compatibility of a model and a property. Relying on the small model property, we prove that this measure can be approximated, within a given error, by using a distance between logical formulas.

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1 Introduction

Many complex natural and man-made systems (e.g., biological, ecological, physical, social, financial, and computational) are modeled as stochastic processes in order to handle either a lack of knowledge or inherent randomness. These systems are frequently studied in interaction with discrete systems, such as controllers, or with interactive environments having continuous behavior. This context has motivated research aiming to develop a general theory of systems able to uniformly treat discrete, continuous and hybrid reactive systems. Two of the central questions of this research are “*when do two systems behave similarly up to some quantifiable observation error?*” and “*is there any (algorithmic) technique to check whether two systems have similar behaviours?*”. These questions are related to the problems of state space reduction (collapsing a model to an equivalent reduced model) and discretization (reduce a continuous or hybrid system to an equivalent discrete one), which are cornerstones in the field of stochastic systems.

In the case of probabilistic systems, *probabilistic bisimulation* [17] has been introduced to relate systems with identical probabilistic behaviours and probabilistic multimodal logic

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(PML) [16, 17, 1, 11, 13] has been used to characterize this equivalence: the logical equivalence induced by PML on probabilistic models coincides with probabilistic bisimulation [17, 20, 10]. However, in spite of the elegant theories supporting it, the concept of bisimulation remains too strict for applications. In modelling, the values of the parameters (rates or probabilities) are often approximated and consequently, one is interested to know whether two processes that differ by a small amount in real-valued parameters show similar (not necessarily identical) behaviours. In such cases, instead of bisimulation relation, one needs a metric to estimate the degree of similarity of two systems in terms of their behaviours.

For quantifying the behavioral similarity of probabilistic systems it has been introduced a class of pseudometrics [21, 6, 20]. In these settings, the distance between two processes is zero iff they are bisimilar; otherwise, they are closer when they differ by a small amount in their probabilistic behaviours. These pseudometrics can be defined on top of PML, as shown in [6, 20], by extending the satisfiability relation $P \Vdash \phi$ to a function d such that $d(P, \phi) \in [0, 1]$ measures the "degree of satisfiability" between the process P and the property ϕ . The function d induces a distance D between processes by $D(P, P') = \sup\{|d(P, \phi) - d(P', \phi)|, \phi \in \mathcal{L}\}$, where \mathcal{L} is the set of logical formulas. However, the computability of D is sometimes problematic, as it is the computability of $d(P, \phi)$ for an infinite or extremely big process P and for this reason approximation techniques such as statistical model checking [15, 22] are used to evaluate $d(P, \phi)$ within a given error.

In this paper we develop and study the *continuous Markovian logic* (CML) which is similar to PML but developed for general stochastic (Markovian) systems. Our models are *continuous-time and continuous-space labelled Markovian processes* (CMPs) [10, 3, 4]. They generalize other probabilistic models such as *labelled Markov processes* [20, 9, 5, 8] and *Harsanyi type spaces* [12, 19]. CML contains modal operators indexed with transition labels a and positive rationals r . The formula $L_r^a \phi$ expresses the fact that the rate of the a -transitions from a given state to the set of states satisfying ϕ is *at least* r ; similarly, $M_r^a \phi$ states that the rate is *at most* r .

In spite of their syntactic similarities, CML and PML are very different. While in the probabilistic case the two modal operators are dual, being related by the De Morgan duality $M_r^a \phi \leftrightarrow L_{1-r}^a \neg \phi$, in the stochastic case they are independent. Moreover, there exists no sound equivalence of type $\neg X_r^a \phi \leftrightarrow Y_s^a \neg \phi$ for $X, Y \in \{L, M\}$, hence no positive normal forms can be defined for CML formulas. This is because the rate of the transitions from a given state m to the set of states satisfying ϕ is not related to the rate of the transitions from m to the set of states satisfying $\neg \phi$. The differences are reflected in the sound-complete axiomatizations that we present both for CML and for its fragment without M_r^a -operators. Many axioms of PML, such as $\vdash L_r^a \top$ or $\vdash L_r^a \phi \rightarrow \neg L_s^a \neg \phi$ for $r + s < 1$ from [23]¹, are not sound for CMPs. Also at the level of the small model property, which in the case of PML relies on the fact that for a fixed integer q there exists a finite number of integers p such that $p/q \in [0, 1]$ (see [23]), a series of nontrivial additional problems rise in the stochastic case.

The construction of a small model for a consistent CML-formula is the cornerstone of this paper supporting not only the weak completeness proofs, but also approximation techniques to evaluate the extension $d(P, \phi) \in [0, 1]$ of satisfiability relation. In the context of a sound and complete axiomatization, one can turn the bisimulation-distance problem, which in the probabilistic case has been addressed semantically, into a syntactic problem centered on provability. Formally, the distance $\bar{d}(\phi, \psi) = \sup\{|d(P, \phi) - d(P, \psi)|, P \in \mathfrak{P}\}$, where \mathfrak{P} is the class of CMPs, measures the similarity between logical formulas in terms of provability: ϕ

¹ The semantics of [23] is in terms of systems where each action is enabled with probability 1.

and ψ are close in \bar{d} if they (or their negations) can be both proved from the same hypothesis. In this context we prove the *strong robustness theorem*: $d(P, \phi) \leq d(P, \psi) + \bar{d}(\phi, \psi)$. In case that \bar{d} is not computable or it is very expensive, one can use our finite model construction to approximate its value. Let $\tilde{d}(\phi, \psi) = \max\{|d(P, \phi) - d(P, \psi)|, P \in \Omega_p[\phi, \psi]\}$, where $\Omega_p[\phi, \psi]$ is the finite model (finite set of processes) constructed for $\phi \wedge \psi$ if it is consistent, or for $\neg(\phi \wedge \psi)$ otherwise, and $p \in \mathbb{N}$ is the parameter involved in the construction. This guarantees the *weak robustness theorem*: $d(P, \phi) \leq d(P, \psi) + \tilde{d}(\phi, \psi) + 2/p$. Using this theorem, one can evaluate $d(P, \phi)$ from the value of $d(P, \psi)$ and this can be used, for instance, in the context of statistical model checking. Of course, the accuracy of this approximation depends on the similarity of ϕ and ψ from a provability perspective, which influences both the distance $\tilde{d}(\phi, \psi)$ and the parameter p of the finite model construction.

To summarize, the achievements of this paper are as follows.

- We introduce Continuous Markovian Logic, a modal logic that expresses quantitative and qualitative properties of continuous Markov processes. CML is endowed with operators that approximate the labelled transition rates of CMPs and allows us to reason on approximated properties. This logic characterizes the stochastic bisimulation of CMPs.
- We present sound and complete Hilbert-style axiomatizations for CMP and for its M_r^a -free fragment. These are very different from the similar probabilistic cases, due to the structural differences between probabilistic and stochastic models and the differences are reflected by the axioms.
- We prove the finite model properties for CML and its restricted fragment. The construction of a finite model for a consistent formula is novel in the way it exploits the granularity and the Archimedean properties of positive rationals.
- We define a distance between logical formulas that corroborates with the distance between a model and a formula proposed in the literature for probabilistic systems. The organization of the space of logical formulas as a pseudometrizable space with a topology sensitive to provability is a novelty in the field of metric semantics. This structure guarantees the strong robustness theorem.
- We show that the complete axiomatization and the finite model construction can be used to approximate the syntactic distance \bar{d} . This idea opens new research perspectives on the direction of designing algorithms to estimate such distances within given errors.

The structure of the paper. The first section establishes some preliminary concepts and notations used in the paper. Section 3 introduces CMPs and their bisimulation. In Section 4 we define the logic CML and in Section 5 we present sound-complete axiomatizations for both CML and its M_r^a -free fragment proving, at the same time, the small model properties. Section 6 introduces the metric semantics and the results related to metrics and bisimulation. The paper also contains a conclusive section where we discuss new research directions deriving from this paper.

2 Preliminary definitions and notations

In this section we introduce some notations and establish the terminology used in the paper.

For arbitrary sets A, B , 2^A denotes the powerset of A and $[A \rightarrow B]$ the set of functions from A to B .

If (M, Σ) is a measurable space with σ -algebra $\Sigma \subseteq 2^M$, we use $\Delta(M, \Sigma)$ to denote the

set of measures² $\mu : \Sigma \rightarrow \mathbb{R}^+$ on (M, Σ) . We organize $\Delta(M, \Sigma)$ as a measurable space by considering the σ -algebra generated, for arbitrary $S \in \Sigma$ and $r > 0$, by the sets

$$\{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}.$$

Given two measurable spaces (M, Σ) and (N, Θ) , we use $\llbracket M \rightarrow N \rrbracket$ to denote the class of measurable mappings from (M, Σ) to (N, Θ) .

Given a relation $\mathfrak{R} \subseteq M \times M$, the set $N \subseteq M$ is \mathfrak{R} -closed iff $\{m \in M \mid \exists n \in N, (n, m) \in \mathfrak{R}\} \subseteq N$. If (M, Σ) is a measurable space and $\mathfrak{R} \subseteq M \times M$, then $\Sigma(\mathfrak{R})$ denotes the set of measurable \mathfrak{R} -closed subsets of M .

3 Continuous Markov processes

Based on an equivalence between the definitions of Harsanyi type spaces [12, 19] and labelled Markov processes [20, 9, 5, 8] evidenced by Doberkat in the light of the Giriy monad [7], we introduce the *continuous Markov processes* (CMPs). CMPs are models of stochastic systems with continuous state space and continuous time transitions. They are defined for a fixed countable set \mathcal{A} of *transition labels* representing the types of interactions with the environment. If $a \in \mathcal{A}$, m is the current state of the system and N is a measurable set of states, the function $\theta(a)(m)$ is a measure on the state space and $\theta(a)(m)(N) \in \mathbb{R}^+$ represents the *rate* of an exponentially distributed random variable that characterizes the duration of an a -transition from m to arbitrary $n \in N$. Indeterminacy in such systems is resolved by races between events executing at different rates.

► **Definition 1** (Continuous Markov processes). Given an analytic set (M, Σ) , where Σ is the Borel algebra generated by the topology, an *\mathcal{A} -continuous Markov kernel* is a tuple $\mathcal{M} = (M, \Sigma, \theta)$, where $\theta : \mathcal{A} \rightarrow \llbracket M \rightarrow \Delta(M, \Sigma) \rrbracket$. M is the support set of \mathcal{M} denoted by $\text{supp}(\mathcal{M})$. If $m \in M$, (\mathcal{M}, m) is an *\mathcal{A} -continuous Markov process*.

Notice that $\theta(a)$ is a measurable mapping between (M, Σ) and $\Delta(M, \Sigma)$. This is equivalent with the conditions on the two-variable *rate function* used in [10] to define continuous Markov processes (for the proof of the equivalence see, e.g. Proposition 2.9, of [7]).

In the rest of the paper we assume that the set of transition labels \mathcal{A} is fixed. We denote by \mathfrak{M} the class of \mathcal{A} -continuous Markov kernels (CMKs) and we use $\mathcal{M}, \mathcal{M}_i, \mathcal{M}'$ to range over \mathfrak{M} . We denote by \mathfrak{P} the set of \mathcal{A} -CMPs and we use P, P_i, P' to range over \mathfrak{P} .

The stochastic bisimulation for CMPs follows the line of Larsen-Skou probabilistic bisimulation [17, 8, 20].

► **Definition 2** (Stochastic Bisimulation). Given $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$, a *rate-bisimulation relation* on \mathcal{M} is a relation $\mathfrak{R} \subseteq M \times M$ such that $(m, n) \in \mathfrak{R}$ iff for any $C \in \Sigma(\mathfrak{R})$ and any $a \in \mathcal{A}$,

$$\theta(a)(m)(C) = \theta(a)(n)(C).$$

Two processes (\mathcal{M}, m) and (\mathcal{M}, n) are *stochastic bisimilar*, written $m \sim_{\mathcal{M}} n$, if they are related by a rate-bisimulation relation.

Observe that, for any $\mathcal{M} \in \mathfrak{M}$ there exist rate-bisimulation relations as, for instance, is the identity relation on \mathcal{M} ; the stochastic bisimulation is the largest rate-bisimulation.

² Notice that in this paper we do not consider infinite rates.

If $\mathcal{M} = (M, \Sigma, \theta)$, $\mathcal{M}' = (M', \Sigma', \theta') \in \mathfrak{M}$, then $\mathcal{M}'' = (M'', \Sigma'', \theta'') = \mathcal{M} \uplus \mathcal{M}'$ if $M'' = M \uplus M'$, Σ'' is generated by $\Sigma \uplus \Sigma'$ and for any $a \in \mathcal{A}$, $N \in \Sigma$ and $N' \in \Sigma'$,

$$\theta''(a)(m)(N \uplus N') = \begin{cases} \theta(a)(m)(N) & \text{if } m \in M \\ \theta'(a)(m)(N') & \text{if } m \in M' \end{cases}$$

Notice that $\mathcal{M}'' \in \mathfrak{M}$. If $m \in M$ and $m' \in M'$, we say that (\mathcal{M}, m) and (\mathcal{M}', m') are *bisimilar* written $(\mathcal{M}, m) \sim (\mathcal{M}', m')$ whenever $m \sim_{\mathcal{M} \uplus \mathcal{M}'} m'$.

4 Continuous Markovian Logics

In this section we introduce the continuous Markovian logic (CML) for semantics based on CMPs. In addition to the Boolean operators, this logic is provided with *stochastic modal operators* that approximate the rates of transitions. For $a \in \mathcal{A}$ and $r \in \mathbb{Q}_+$, $L_r^a \phi$ characterizes (\mathcal{M}, m) whenever the rate of the a -transition from m to the class of the states satisfying ϕ is *at least* r ; symmetrically, $M_r^a \phi$ is satisfied when this rate is *at most* r . CMLs extends the probabilistic logics [1, 16, 13, 23, 11] to stochastic domains. The obvious structural similarities between the probabilistic and the stochastic models are not preserved when we consider the logic. By focusing on general measures instead of probabilistic measures in the definition of the transition systems, many of the axioms of probabilistic logics are not sound for stochastic semantics. This is the case, for instance, with $\vdash L_r^a \top$ or $\vdash L_r^a \phi \rightarrow \neg L_s^a \neg \phi$ for $r + s < 1$ which are proposed in [11]. Moreover, while in probabilistic settings the operators L_r^a and M_s^a are dual, satisfying $M_r^a \phi = L_{1-r}^a \neg \phi$, they became independent in stochastic semantics. For this reason, in the next section we study two CML logics with complete axiomatizations, \mathcal{L} involving only the stochastic operators of type L_r^a and \mathcal{L}^+ that contains both L_r^a and M_s^a .

► **Definition 3** (Syntax). Given a countable set \mathcal{A} , the formulas of $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^+(\mathcal{A})$ respectively are introduced by the following grammars, for arbitrary $a \in \mathcal{A}$ and $r \in \mathbb{Q}_+$.

$$\mathcal{L}(\mathcal{A}) : \quad \phi := \top \mid \neg \phi \mid \phi \wedge \psi \mid L_r^a \phi,$$

$$\mathcal{L}^+(\mathcal{A}) : \quad \phi := \top \mid \neg \phi \mid \phi \wedge \psi \mid L_r^a \phi \mid M_r^a \phi.$$

In addition, we assume all the Boolean operators and $\perp = \neg \top$, as well as the derived operator $E_r^a \phi = L_r^a \phi \wedge M_r^a \phi$.

In what follows we use the same set \mathcal{A} of labels used with CMPs. The semantics of $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^+(\mathcal{A})$, called in this paper *Markovian semantics*, are defined by the *satisfiability relation* for arbitrary \mathcal{A} -CMPs (\mathcal{M}, m) with $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$, by:

- $\mathcal{M}, m \Vdash \top$ always,
- $\mathcal{M}, m \Vdash \neg \phi$ iff it is not the case that $\mathcal{M}, m \Vdash \phi$,
- $\mathcal{M}, m \Vdash \phi \wedge \psi$ iff $\mathcal{M}, m \Vdash \phi$ and $\mathcal{M}, m \Vdash \psi$,
- $\mathcal{M}, m \Vdash L_r^a \phi$ iff $\theta(a)(m)(\llbracket \phi \rrbracket_{\mathcal{M}}) \geq r$,
- $\mathcal{M}, m \Vdash M_r^a \phi$ iff $\theta(a)(m)(\llbracket \phi \rrbracket_{\mathcal{M}}) \leq r$,

where $\llbracket \phi \rrbracket_{\mathcal{M}} = \{m \in M \mid \mathcal{M}, m \Vdash \phi\}$.

When it is not the case that $\mathcal{M}, m \Vdash \phi$, we write $\mathcal{M}, m \not\Vdash \phi$.

We have that $\mathcal{M}, m \not\Vdash \perp$ always and that $\mathcal{M}, m \Vdash E_r^a \phi$ iff $\theta(a)(m)(\llbracket \phi \rrbracket_{\mathcal{M}}) = r$. Notice that $E_r^a \phi$ characterizes the process that can do an a -transition to the set of processes satisfying ϕ with the rate r . So, in this case one can express the exact rate of the transitions. This is always possible in probabilistic logic where M_r^a and L_r^a are dual operators and consequently E_r^a is always definable. In the stochastic case L_r^a , M_r^a and E_r^a are mutually independent. We

chose not to study a Markovian logic that involves only the E_r^a operators because in many applications we do not know the exact rates of the transitions and it is more useful to work with approximations such as M_r^a or L_r^a .

The semantics of $L_r^a\phi$ and $M_r^a\phi$ are well defined only if $\llbracket\phi\rrbracket_{\mathcal{M}}$ is measurable. This is guaranteed by the fact that $\theta(a)$ is a measurable mapping between (M, Σ) and $\Delta(M, \Sigma)$, as proved in the next lemma.

► **Lemma 4.** *For any $\phi \in \mathcal{L}^+(\mathcal{A})$ and any $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$, $\llbracket\phi\rrbracket_{\mathcal{M}} \in \Sigma$.*

Proof. Induction on ϕ : for $\phi = L_r^a\psi$, the inductive hypothesis guarantees that $\llbracket\psi\rrbracket_{\mathcal{M}} \in \Sigma$, hence, $\{\mu \in \Delta(M, \Sigma) \mid \mu(\llbracket\psi\rrbracket_{\mathcal{M}}) \geq r\}$ is measurable in $\Delta(M, \Sigma)$. Because $\theta(a)$ is a measurable mapping, we obtain that $\llbracket L_r^a\psi \rrbracket_{\mathcal{M}} = (\theta(a))^{-1}(\{\mu \in \Delta(M, \Sigma) \mid \mu(\llbracket\psi\rrbracket_{\mathcal{M}}) \geq r\})$ is measurable. Similarly it can be proved for $\phi = M_r^a\psi$. ◀

A formula ϕ is *satisfiable* if there exists $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$ and $m \in M$ such that $\mathcal{M}, m \Vdash \phi$. ϕ is *valid*, denoted by $\Vdash \phi$, if $\neg\phi$ is not satisfiable.

5 Complete axiomatizations

In this section we present two Hilbert-style axiomatizations, one for $\mathcal{L}(\mathcal{A})$ and one for $\mathcal{L}^+(\mathcal{A})$, and we prove that they are sound and (weak) complete against the Markovian semantics. Both axiomatizations, as in the case of the axiomatization proposed in [23] for probabilistic systems, contain infinitary rules that encode the Archimedean properties of $\mathbb{Q}_+ \cup \{+\infty\}$. However, as has been shown in [14] following the lines of [13], a finitary axiomatic system can be given at the price of replacing the stochastic operators with some more complex operators. For our purpose of reasoning on approximated properties of Markovian processes, a complete axiomatization involving only the stochastic operators (and their Archimedean rules) is more useful.

5.1 Axiomatization for $\mathcal{L}(\mathcal{A})$

Table 1 contains a Hilbert-style axiomatization for $\mathcal{L}(\mathcal{A})$. The axioms and rules, considered in addition to the axiomatization of classic propositional logic, are given for propositional variables $\phi, \psi \in \mathcal{L}(\mathcal{A})$, for arbitrary $a \in \mathcal{A}$ and $s, r \in \mathbb{Q}^+$.

- (A1): $\vdash L_0^a\phi$
- (A2): $\vdash L_{r+s}^a\phi \rightarrow L_r^a\phi$
- (A3): $\vdash L_r^a(\phi \wedge \psi) \wedge L_s^a(\phi \wedge \neg\psi) \rightarrow L_{r+s}^a\phi$
- (A4): $\vdash \neg L_r^a(\phi \wedge \psi) \wedge \neg L_s^a(\phi \wedge \neg\psi) \rightarrow \neg L_{r+s}^a\phi$
- (R1): If $\vdash \phi \rightarrow \psi$ then $\vdash L_r^a\phi \rightarrow L_r^a\psi$
- (R2): If $\forall r < s, \vdash \phi \rightarrow L_r^a\psi$ then $\vdash \phi \rightarrow L_s^a\psi$
- (R3): If $\forall r > s, \vdash \phi \rightarrow L_r^a\psi$ then $\vdash \phi \rightarrow \perp$

■ **Table 1** The axiomatic system of $\mathcal{L}(\mathcal{A})$

This axiomatic system has some similarities to the axiomatic system of probabilistic logic proposed in [23] for Harsanyi type spaces. The main difference is that the axioms of probabilistic logic $\vdash L_r^a\top$ and $\vdash L_r^a\phi \rightarrow \neg L_s^a\neg\phi$ for $r + s \leq 1$ are not sound for the Markovian semantics and this changes the entire proof structure. We also have two Archimedean

properties reflected in (R2) and (R3); while the first allows us to argue on convergent sequences of rationals, the second excludes the models with infinite rates.

As usual, we say that a formula ϕ is *provable*, denoted by $\vdash \phi$, if it can be proved from the given axioms and rules. We say that ϕ is *consistent*, if $\phi \rightarrow \perp$ is not provable. Given a set Φ of formulas, we say that Φ proves ϕ , $\Phi \vdash \phi$, if from the formulas of Φ and the axioms one can prove ϕ . Φ is *consistent* if it is not the case that $\Phi \vdash \perp$. For a sublanguage $\mathcal{L} \subseteq \mathcal{L}^+(\mathcal{A})$, we say that Φ is *\mathcal{L} -maximally consistent* if Φ is consistent and no formula of \mathcal{L} can be added to it without making it inconsistent.

► **Theorem 5 (Soundness).** *The axiomatic system of $\mathcal{L}(\mathcal{A})$ is sound for the Markovian semantics, i.e., for any $\phi \in \mathcal{L}(\mathcal{A})$, if $\vdash \phi$ then $\Vdash \phi$.*

In what follows we prove the finite model property for $\mathcal{L}(\mathcal{A})$ using the filtration method adapted for CMPs. This result will eventually establish the (weak) completeness of the axiomatic system for the Markovian semantics, meaning that everything that is true for all the models is also provable. This logic is not complete because the stochastic operators are not compact.

To prove the weak completeness we will construct, for an arbitrary consistent formula $\psi \in \mathcal{L}(\mathcal{A})$, a model $(\mathcal{M}_\psi, \Gamma)$ where $\text{supp}(\mathcal{M}_\psi)$ is a finite set of $\mathcal{L}(\mathcal{A})$ -consistent sets of formulas. As usual with the filtration method, the key argument is the truth lemma: $\psi \in \Gamma$ iff $\mathcal{M}_\psi, \Gamma \Vdash \psi$. A similar construction has been proposed in [23] for probabilistic logic, where the finite model property derives from the fact that the number of rationals of type $\frac{p}{n}$, for a fixed integer n , is finite within $[0, 1]$. The same property does not hold in our case, as the focus is on $[0, \infty)$, and instead we need a more complicated construction.

Before proceeding with the construction, we fix some notations.

For $n \in \mathbb{N}$, $n \neq 0$, let $\mathbb{Q}_n = \{\frac{p}{n} : p \in \mathbb{N}\}$. If $S \subseteq \mathbb{Q}$ is finite, the *granularity* of S , $gr(S)$, is the least common multiple of the denominators of the elements of S .

The *modal depth* of $\phi \in \mathcal{L}(\mathcal{A})$ is defined by $md(\top) = 0$, $md(\neg\phi) = md(\phi)$, $md(\phi \wedge \psi) = \max(md(\phi), md(\psi))$ and $md(L_r^a \phi) = md(\phi) + 1$.

The *granularity* of $\phi \in \mathcal{L}$ is $gr(\phi) = gr(R)$, where $R \subseteq \mathbb{Q}_+$ is the set of indexes r of the operators L_r^a present in ϕ ; the *upper bound* of ϕ is $\max(\phi) = \max(R)$.

The *actions* of ϕ is the set $act(\phi) \subseteq \mathcal{A}$ of indexes $a \in \mathcal{A}$ of the operators L_r^a present in ϕ .

For arbitrary $n \in \mathbb{N}$ and $A \subseteq \mathcal{A}$, let $\mathcal{L}_n(A)$ be the sublanguage of $\mathcal{L}(\mathcal{A})$ that uses only modal operators L_r^a with $r \in \mathbb{Q}_n$ and $a \in A$. For $\Lambda \subseteq \mathcal{L}(\mathcal{A})$, let $[\Lambda]_n = \Lambda \cup \{\phi \in \mathcal{L}_n(\mathcal{A}) : \Lambda \vdash \phi\}$.

Consider a consistent formula $\psi \in \mathcal{L}(\mathcal{A})$ with $gr(\psi) = n$ and $act(\psi) = A$.

Let $\mathcal{L}[\psi] = \{\phi \in \mathcal{L}_n(A) \mid \max(\phi) \leq \max(\psi), md(\phi) \leq md(\psi)\}$.

In what follows we construct $\mathcal{M}_\psi \in \mathfrak{M}$ such that each $\Gamma \in \text{supp}(\mathcal{M}_\psi)$ is a consistent set of formulas that contains an $\mathcal{L}[\psi]$ -maximally consistent set of formulas and each $\mathcal{L}[\psi]$ -maximally consistent set is contained in some $\Gamma \in \text{supp}(\mathcal{M}_\psi)$. And we will prove that for $\phi \in \mathcal{L}[\psi]$, $\phi \in \Gamma$ iff $\mathcal{M}_\psi, \Gamma \Vdash \phi$.

Let $\Omega[\psi]$ be the set of $\mathcal{L}[\psi]$ -maximally consistent sets of formulas. $\Omega[\psi]$ is finite and any $\Lambda \in \Omega[\psi]$ contains finitely many *nontrivial formulas*³; in the rest of this construction we only count non-trivial formulas while ignoring the rest and we use $\bigwedge \Lambda$ to denote the conjunction of the nontrivial formulas of Λ .

³ By nontrivial formulas we mean the formulas that are not obtained from more basic consistent ones by boolean derivations. For instance $p \vee q \rightarrow p$, $p \wedge p$, $p \vee p$ are trivial formulas.

For each $\Lambda \in \Omega[\psi]$, such that $\{\phi_1, \dots, \phi_i\} \subseteq \Lambda$ is its set of its non-trivial formulas, we construct $\Lambda^+ \supseteq [\Lambda]_n$ with the property that $\forall \phi \in \Lambda$ and $a \in \mathcal{A}$ there exists $\neg L_r^a \phi \in \Lambda^+$.

The construction step $[\phi_1$ versus Λ]:

(R3) guarantees that $\exists r \in \mathbb{Q}_n$ s.t. $[\Lambda]_n \cup \{\neg L_r^a \phi_1\}$ is consistent (suppose that this is not the case, then $\vdash \bigwedge \Lambda \rightarrow L_r^a \phi_1$ for all $r \in \mathbb{Q}_n$ implying that $\bigwedge \Lambda$ inconsistent - impossible). Let $y_1^a = \min\{s \in \mathbb{Q}_n : [\Lambda]_n \cup \{\neg L_s^a \phi_1\} \text{ is consistent}\}$ and $x_1^a = \max\{s \in \mathbb{Q}_n : L_s^a \phi_1 \in [\Lambda]_n\}$ ((R3) guarantees the existence of max, because otherwise $\vdash \bigwedge \Lambda \rightarrow L_r^a \phi_1$ for all r implying $\bigwedge \Lambda$ inconsistent - impossible). (R2) implies that $\exists r \in \mathbb{Q} \setminus \mathbb{Q}_n$ s.t., $x_1^a < r < y_1^a$ and $\{\neg L_r^a \phi_1\} \cup [\Lambda]_n$ is consistent (otherwise, $\vdash \bigwedge \Lambda \rightarrow L_r^a \phi_1$ for all $r < y_1^a$ and due to (R2), $\vdash \bigwedge \Lambda \rightarrow L_{y_1^a}^a \phi_1$ - contradiction with the consistency of Λ). Obviously, $r \notin \mathbb{Q}_n$. Let $n_1 = \text{gran}\{1/n, r\}$. Let $s_1^a = \min\{s \in \mathbb{Q}_{n_1} : [\Lambda]_{n_1} \cup \{\neg L_s^a \phi_1\} \text{ is consistent}\}$, $\Lambda_1^a = \Lambda \cup \{\neg L_{s_1^a}^a \phi_1\}$ and $\Lambda_1 = \bigcup_{a \in A} \Lambda_1^a$.

The construction step $[\phi_2$ versus Λ_1]:

As before, let $y_2^a = \min\{s \in \mathbb{Q}_{n_1} : [\Lambda_1]_{n_1} \cup \{\neg L_s^a \phi_2\} \text{ is consistent}\}$ and $x_2^a = \max\{s \in \mathbb{Q}_{n_1} : L_s^a \phi_2 \in [\Lambda_1]_{n_1}\}$. There exists $r \in \mathbb{Q} \setminus \mathbb{Q}_{n_1}$ s.t., $x_2^a < r < y_2^a$ and $\{\neg L_r^a \phi_2\} \cup [\Lambda_1]_{n_1}$ is consistent. Let $n_2 = \text{gran}\{1/n_1, r\}$. Let $s_2^a = \min\{s \in \mathbb{Q}_{n_2} : [\Lambda]_{n_2} \cup \{\neg L_s^a \phi_2\} \text{ is consistent}\}$, $\Lambda_2^a = \Lambda_1 \cup \{\neg L_{s_2^a}^a \phi_2\}$ and $\Lambda_2 = \bigcup_{a \in A} \Lambda_2^a$.

We repeat this construction step for $[\phi_3$ versus $\Lambda_2], \dots, [\phi_i$ versus $\Lambda_{i-1}]$ and in a finite number of steps we eventually obtain $\Lambda \subseteq \Lambda_1 \subseteq \dots \subseteq \Lambda_i$, where Λ_i is a consistent set containing a finite set of nontrivial formulas. Let $n_\Lambda = \text{gran}\{1/n_1, \dots, 1/n_i\}$. We make this construction for all $\Lambda \in \Omega[\psi]$. Let $p = \text{gran}\{1/n_\Lambda : \Lambda \in \Omega[\psi]\}$. Notice that $p > n$. Let $\Lambda^+ = [\Lambda_i]_p$ and $\Omega^+[\psi] = \{\Lambda^+ : \Lambda \in \Omega[\psi]\}$.

► **Remark.** Any consistent formula $\phi \in \mathcal{L}[\psi]$ is an element of a set $\Lambda^+ \in \Omega^+[\psi]$. For each $\Lambda \in \Omega[\psi]$, each $\phi \in \Lambda$ and $a \in \mathcal{A}$, there exist $s, t \in \mathbb{Q}_p$, $s < t$, such that $L_s^a \phi, \neg L_t^a \phi \in \Lambda^+$. Moreover, for any Λ^+ there exists a formula ρ such that $\phi \in \Lambda^+$ iff $\vdash \rho \rightarrow \phi$.

Let Ω_p be the set of $\mathcal{L}_p(\mathcal{A})$ -maximally consistent sets of formulas. We fix an injective (choice) function $f : \Omega^+[\psi] \rightarrow \Omega_p$ such that for any $\Lambda^+ \in \Omega^+[\psi]$, $\Lambda^+ \subseteq f(\Lambda^+)$. We denote by $\Omega_p[\psi] = f(\Omega^+[\psi])$. For $\phi \in \mathcal{L}[\psi]$, let $\llbracket \phi \rrbracket = \{\Gamma \in \Omega_p[\psi] : \phi \in \Gamma\}$. Anticipating the further construction, we will use $\Omega_p[\psi]$ as the support-set for \mathcal{M}_ψ . For this reason we establish some properties for this set.

- **Lemma 6. 1.** $\Omega_p[\psi]$ is finite.
- 2. $2^{\Omega_p[\psi]} = \{\llbracket \phi \rrbracket : \phi \in \mathcal{L}[\psi]\}$.
- 3. For any $\phi_1, \phi_2 \in \mathcal{L}[\psi]$, $\vdash \phi_1 \rightarrow \phi_2$ iff $\llbracket \phi_1 \rrbracket \subseteq \llbracket \phi_2 \rrbracket$.
- 4. For any $\Gamma \in \Omega_p[\psi]$, $\phi \in \mathcal{L}[\psi]$ and $a \in A$ there exist $x = \max\{r \in \mathbb{Q}_p : L_r^a \phi \in \Gamma\}$, $y = \min\{r \in \mathbb{Q}_p : \neg L_r^a \phi \in \Gamma\}$ and $y = x + 1/p$.

Proof. 4. $L_x^a \phi, \neg L_y^a \phi \in \Gamma$ implies $x \neq y$. If $x > y$, $L_x^a \phi \in \Gamma$ entails (Axiom (A2)) $L_y^a \phi \in \Gamma$, contradicting the consistency of Γ . If $x + 1/p < y$, then $L_{x+1/p}^a \phi \notin \Gamma$, i.e. $\neg L_{x+1/p}^a \phi \in \Gamma$ implying that $x + 1/p \geq y$ - contradiction. ◀

Let Ω be the set of $\mathcal{L}(\mathcal{A})$ -maximally consistent sets of formulas. We fix an injective (choice) function $g : \Omega_p \rightarrow \Omega$ such that for any $\Gamma \in \Omega_p$, $\Gamma \subseteq \pi(\Gamma)$; we denote $g(\Gamma)$ by Γ^∞ .

► **Lemma 7.** For any $\Gamma \in \Omega_p[\psi]$, $\phi \in \mathcal{L}[\psi]$ and $a \in A$, there exists

$$z = \sup\{r \in \mathbb{Q} : L_r^a \phi \in \Gamma^\infty\} = \inf\{r \in \mathbb{Q} : \neg L_r^a \phi \in \Gamma^\infty\} \text{ and } x \leq z < y.$$

Proof. Let $x^\infty = \sup\{r \in \mathbb{Q} : L_r^a \phi \in \Gamma^\infty\}$ and $y^\infty = \inf\{r \in \mathbb{Q} : \neg L_r^a \phi \in \Gamma^\infty\}$. Suppose that $x^\infty < y^\infty$. Then there exists $r \in \mathbb{Q}$ such that $x^\infty < r < y^\infty$. Hence, $\neg L_r^a \phi, L_r^a \phi \in \Gamma^\infty$ -

impossible because Γ^∞ is consistent. Suppose that $x^\infty > y^\infty$. Then there exists $r \in \mathbb{Q}$ such that $x^\infty > r > y^\infty$. As Γ^∞ is maximally consistent we have either $L_r^a \phi \in \Gamma^\infty$ or $\neg L_r^a \phi \in \Gamma^\infty$. The first case contradicts the definition of x^∞ while the second the definition of y^∞ . Hence, $x \leq z \leq y$. If $z = y$, then $L_z^a \phi, \neg L_z^a \phi \in \Gamma$ contradicting the consistency of Γ . ◀

We denote z by a_ϕ^Γ and now we can define \mathcal{M}_ψ .

► **Lemma 8.** *If $\theta_\psi : \mathcal{A} \rightarrow [\Omega_p[\psi] \rightarrow \Delta(\Omega_p[\psi], 2^{\Omega_p[\psi]})]$ is defined for arbitrary $a \in \mathcal{A}$, $\Gamma \in \Omega_q[\psi]$ and $\phi \in \mathcal{L}[\psi]$ by $\theta_\psi(a)(\Gamma)(\llbracket \phi \rrbracket) = a_\phi^\Gamma$, then $\mathcal{M}_\psi = (\Omega_p[\psi], 2^{\Omega_p[\psi]}, \theta_\psi) \in \mathfrak{M}$.*

Proof. The central problem is to prove that for arbitrary $\Gamma \in \Omega_p[\psi]$ and $a \in \mathcal{A}$, the function $\theta_\psi(a)(\Gamma) : 2^{\Omega_p[\psi]} \rightarrow \mathbb{R}^+$ is well defined and a measure on $(\Omega_p[\psi], 2^{\Omega_p[\psi]})$. Further, because the space is discrete with finite support, we obtain that $\theta_\psi(a) \in \llbracket \Omega_p[\psi] \rightarrow \Delta(\Omega_p[\psi], 2^{\Omega_p[\psi]}) \rrbracket$.

Suppose that for $\phi_1, \phi_2 \in \mathcal{L}[\psi]$ we have $\llbracket \phi_1 \rrbracket = \llbracket \phi_2 \rrbracket$. Then, from Lemma 6, $\vdash \phi_1 \leftrightarrow \phi_2$ and $\vdash L_r^a \phi_1 \leftrightarrow L_r^a \phi_2$. Hence, $a_{\phi_1}^\Gamma = a_{\phi_2}^\Gamma$ proving that $\theta_\psi(a)(\Gamma)$ is well defined.

Now we prove that $\theta_\psi(a)(\Gamma)$ is a measure.

For showing $\theta_\psi(a)(\Gamma)(\emptyset) = 0$, we show that for any $r > 0$, $\vdash \neg L_r^a \perp$. This is sufficient, as (A1) guarantees that $\vdash L_0^a \perp$ and $\llbracket \perp \rrbracket = \emptyset$. Suppose that there exists $r > 0$ such that $L_r^a \perp$ is consistent. Let $\epsilon \in (0, r) \cap \mathbb{Q}$. Then (A2) gives $\vdash L_r^a \perp \rightarrow L_\epsilon^a \perp$. Hence, $\vdash L_r^a \perp \rightarrow (L_r^a(\perp \wedge \perp) \wedge L_\epsilon^a(\perp \wedge \neg \perp))$ and applying (A3), $\vdash L_r^a \perp \rightarrow L_{r+\epsilon}^a \perp$. Repeating this argument, we can prove that $\vdash L_r^a \perp \rightarrow L_s^a \perp$ for any s and (R3) confirms the inconsistency of $L_r^a \perp$.

We show now that if $A, B \in 2^{\Omega_p[\psi]}$ with $A \cap B = \emptyset$, then $\theta_\psi(a)(\Gamma)(A) + \theta_\psi(a)(\Gamma)(B) = \theta_\psi(a)(\Gamma)(A \cup B)$. Let $A = \llbracket \phi_1 \rrbracket$, $B = \llbracket \phi_2 \rrbracket$ with $\phi_1, \phi_2 \in \mathcal{L}[\psi]$ and $\vdash \phi_1 \rightarrow \neg \phi_2$. Let $x_1 = \theta_\psi(a)(\Gamma)(A)$, $x_2 = \theta_\psi(a)(\Gamma)(B)$ and $x = \theta_\psi(a)(\Gamma)(A \cup B)$. We prove that $x_1 + x_2 = x$.

Suppose that $x_1 + x_2 < x$. Then, there exist $\epsilon_1, \epsilon_2 \in \mathbb{Q}^+$ such that $x'_1 + x'_2 < x$, where $x'_i = x_i + \epsilon_i$ for $i = 1, 2$. From the definition of x_i , $\neg L_{x'_i}^a \phi_i \in \Gamma^\infty$. Further, using (A4), we obtain $\neg L_{x'_1+x'_2}^a(\phi_1 \vee \phi_2) \in \Gamma^\infty$, implying that $x'_1 + x'_2 \geq x$ - contradiction.

Suppose that $x_1 + x_2 > x$. Then, there exist $\epsilon_1, \epsilon_2 \in \mathbb{Q}^+$ such that $x''_1 + x''_2 > x$, where $x''_i = x_i - \epsilon_i$ for $i = 1, 2$. But the definition of x_i implies that $L_{x''_i}^a \phi_i \in \Gamma^\infty$. Further, (A3) gives $L_{x''_1+x''_2}^a(\phi_1 \vee \phi_2) \in \Gamma^\infty$, i.e. $x''_1 + x''_2 \leq x$ - contradiction. ◀

Now we can prove the Truth Lemma.

► **Lemma 9 (Truth Lemma).** *If $\phi \in \mathcal{L}[\psi]$, then $[\mathcal{M}_\psi, \Gamma \Vdash \phi \text{ iff } \phi \in \Gamma]$.*

Proof. Induction on the structure of ϕ . The only nontrivial case is $\phi = L_r^a \phi'$.

(\implies) Suppose that $\mathcal{M}_\psi, \Gamma \Vdash \phi$ and $\phi \notin \Gamma$. Hence $\neg \phi \in \Gamma$. Let $y = \min\{r \in \mathbb{Q}_p : \neg L_r^a \phi \in \Gamma\}$. Then, from $\neg L_r^a \phi \in \Gamma$, we obtain $r \geq y$. But $\mathcal{M}_\psi, \Gamma \Vdash L_r^a \phi'$ is equivalent with $\theta_\psi(a)(\Gamma)(\llbracket \phi' \rrbracket) \geq r$, i.e. $a_{\phi'}^\Gamma \geq r$. On the other hand, from Lemma 6, $a_{\phi'}^\Gamma < y$ - contradiction.
(\impliedby) If $L_r^a \phi' \in \Gamma$, then $r \leq a_{\phi'}^\Gamma$ and $r \leq \theta_\psi(a)(\Gamma)(\llbracket \phi' \rrbracket)$. Hence, $\mathcal{M}_\psi, \Gamma \Vdash L_r^a \phi$. ◀

The previous lemma implies the small model property for our logic.

► **Theorem 10 (Small model property).** *For any $\mathcal{L}(\mathcal{A})$ -consistent formula ϕ , there exists $\mathcal{M} \in \mathfrak{M}$ with finite support of cardinality bound by the structure of ϕ , and there exists $m \in \text{supp}(\mathcal{M})$ such that $\mathcal{M}, m \Vdash \phi$.*

The small model property proves the (weak) completeness of the axiomatic system.

► **Theorem 11 (Weak Completeness).** *The axiomatic system of $\mathcal{L}(\mathcal{A})$ is complete with respect to the Markovian semantics, i.e. if $\vdash \psi$, then $\vdash \psi$.*

Proof. We have that $[\Vdash \psi \text{ implies } \vdash \psi]$ is equivalent with $[\not\vdash \psi \text{ implies } \not\vdash \psi]$, that is equivalent with [the consistency of $\neg\psi$ implies the existence of a model (\mathcal{M}, m) for $\neg\psi$] and this is guaranteed by the finite model property. \blacktriangleleft

5.2 Axiomatization for $\mathcal{L}^+(\mathcal{A})$

Table 2 contains a Hilbert-style axiomatization for $\mathcal{L}^+(\mathcal{A})$.

- (B1): $\vdash L_0^a \phi$
- (B2): $\vdash L_{r+s}^a \phi \rightarrow \neg M_r^a \phi, s > 0$
- (B3): $\vdash \neg L_r^a \phi \rightarrow M_r^a \phi$
- (B4): $\vdash \neg L_r^a(\phi \wedge \psi) \wedge \neg L_s^a(\phi \wedge \neg\psi) \rightarrow \neg L_{r+s}^a \phi$
- (B5): $\vdash \neg M_r^a(\phi \wedge \psi) \wedge \neg M_s^a(\phi \wedge \neg\psi) \rightarrow \neg M_{r+s}^a \phi$
- (S1): If $\vdash \phi \rightarrow \psi$ then $\vdash L_r^a \phi \rightarrow L_r^a \psi$
- (S2): If $\forall r < s, \vdash \phi \rightarrow L_r^a \psi$ then $\vdash \phi \rightarrow L_s^a \psi$
- (S3): If $\forall r > s, \vdash \phi \rightarrow M_r^a \psi$ then $\vdash \phi \rightarrow M_s^a \psi$
- (S4): If $\forall r > s, \vdash \phi \rightarrow L_r^a \psi$ then $\vdash \phi \rightarrow \perp$

■ **Table 2** The axiomatic system of $\mathcal{L}^+(\mathcal{A})$

Notice the differences between these axioms and the axioms in Table 1. First of all Axiom (A2) had to be enforced by (B2) and (B3) which depict the connection between the two stochastic operators. In the probabilistic case these relations are encoded by the duality rule $M_r^a \phi = L_{1-r}^a \neg\phi$ and by the axiom $\vdash L_r^a \phi \rightarrow \neg L_s^a \neg\phi$ for $r + s < 1$; these two are not sound for stochastic models. Rule (A3) has been itself enforced by (B5). We also have an extra Archimedean rule for M_r^a . We prove below that all the theorems of $\mathcal{L}(\mathcal{A})$ are also theorems of $\mathcal{L}^+(\mathcal{A})$ and we state some theorems of $\mathcal{L}^+(\mathcal{A})$ that are central for the weak completeness proof of $\mathcal{L}^+(\mathcal{A})$.

- **Lemma 12.** 1. $\vdash L_{r+s}^a \phi \rightarrow L_r^a \phi, \quad 2. \vdash M_r^a \phi \rightarrow M_{r+s}^a \phi,$
 3. $\vdash L_r^a(\phi \wedge \psi) \wedge L_s^a(\phi \wedge \neg\psi) \rightarrow L_{r+s}^a \phi, \quad 4. \vdash M_r^a(\phi \wedge \psi) \wedge M_s^a(\phi \wedge \neg\psi) \rightarrow M_{r+s}^a \phi,$
 5. $\vdash \neg M_r^a \phi \rightarrow L_r^a \phi, \quad 6. \vdash M_r^a \phi \rightarrow \neg L_{r+s}^a \phi, s > 0,$
 7. If $\vdash \phi \rightarrow \psi$, then $\vdash M_r^a \psi \rightarrow M_r^a \phi$.

► **Theorem 13 (Soundness).** *The axiomatic system of $\mathcal{L}^+(\mathcal{A})$ is sound for the Markovian semantics, i.e., for any $\phi \in \mathcal{L}^+(\mathcal{A})$, if $\vdash \phi$ then $\Vdash \phi$.*

The finite model property for $\mathcal{L}^+(\mathcal{A})$ is proved, similarly to the case of $\mathcal{L}(\mathcal{A})$, by using the filtration method. In what follows we will not reproduce the entire construction already presented for $\mathcal{L}(\mathcal{A})$, but we only emphasize the major differences.

We keep the notations introduced before with the only differences that for an arbitrary $\phi \in \mathcal{L}^+(\mathcal{A})$, the definition of the modal depth of ϕ also includes $md(M_r^a \psi) = md(\psi) + 1$ and $gr(\phi)$, $max(\phi)$ and $act(\phi)$ take into account, in addition, the indexes of the operators of type M_r^a that appear in ϕ . With these modifications, we define $\mathcal{L}_n^+(\mathcal{A})$, for any integer n and $A \subseteq \mathcal{A}$, as before and for $\Lambda \subseteq \mathcal{L}^+(\mathcal{A})$, $[\Lambda]_n = \Lambda \cup \{\phi \in \mathcal{L}_n^+(\mathcal{A}) : \Lambda \vdash \phi\}$.

Consider a consistent formula $\psi \in \mathcal{L}^+(\mathcal{A})$ with $gr(\psi) = n$ and $act(\psi) = A$. We define $\mathcal{L}^+[\psi] = \{\phi \in \mathcal{L}_n^+(\mathcal{A}) \mid max(\phi) \leq max(\psi), md(\phi) \leq md(\psi)\}$. Let $\Omega[\psi]$ be the set of $\mathcal{L}^+[\psi]$ -maximally consistent sets of formulas. We remake the construction done in the previous subsection for $\mathcal{L}(\mathcal{A})$.

The first important difference with respect to the previous case appears due to (B2): for each $\Lambda \in \Omega[\psi]$, $\phi \in \Lambda$ and $a \in \mathcal{A}$, there exist $s, t \in \mathbb{Q}_p$, $s < t$, such that $L_s^a \phi, M_r^a \phi \in \Lambda^+$.

Secondly, for any $\Gamma \in \Omega_p[\psi]$, $\phi \in \mathcal{L}^+[\psi]$ and $a \in A$, there exist $x = \max\{r \in \mathbb{Q}_p : \neg M_r^a \phi \in \Gamma\}$, $y = \min\{r \in \mathbb{Q}_p : M_r^a \phi \in \Gamma\}$ and $y = x + 1/p$. In effect, in the correspondent of Lemma 7, one can prove that there exists $z = \sup\{r \in \mathbb{Q} : L_r^a \phi \in \Gamma^\infty\} = \inf\{r \in \mathbb{Q} : \neg L_r^a \phi \in \Gamma^\infty\} = \inf\{r \in \mathbb{Q} : M_r^a \phi \in \Gamma^\infty\} = \sup\{r \in \mathbb{Q} : \neg M_r^a \phi \in \Gamma^\infty\}$.

As before, we denote z by a_ϕ^Γ and we proceed with the definition of the model \mathcal{M}_ψ .

► **Lemma 14.** *If $\theta_\psi : \mathcal{A} \rightarrow [\Omega_p[\psi] \rightarrow \Delta(\Omega_p[\psi], 2^{\Omega_p[\psi]})]$ is defined for arbitrary $a \in \mathcal{A}$, $\Gamma \in \Omega_q[\psi]$ and $\phi \in \mathcal{L}^+[\psi]$ by $\theta_\psi(a)(\Gamma)(\llbracket \phi \rrbracket) = a_\phi^\Gamma$, then $\mathcal{M}_\psi = (\Omega_p[\psi], 2^{\Omega_p[\psi]}, \theta_\psi) \in \mathfrak{M}$.*

This last result allows us to prove the Truth Lemma for $\mathcal{L}^+(\mathcal{A})$.

► **Lemma 15 (Truth Lemma).** *If $\phi \in \mathcal{L}^+[\psi]$, then $[\mathcal{M}_\psi, \Gamma \Vdash \phi$ iff $\phi \in \Gamma$].*

The proof of Lemma 15 requires, in addition to the proof of Lemma 9, the case $\phi = M_r^a \phi'$ which is proved similarly to the case $\phi = L_r^a \phi'$.

As before, the truth lemma implies the finite model property and the weak completeness theorem for $\mathcal{L}^+(\mathcal{A})$ with Markovian semantics.

► **Theorem 16 (Small model property).** *For any $\mathcal{L}^+(\mathcal{A})$ -consistent formula ϕ , there exists $\mathcal{M} \in \mathfrak{M}$ with finite support of cardinality bound by the structure of ϕ , and there exists $m \in \text{supp}(\mathcal{M})$ such that $\mathcal{M}, m \Vdash \phi$.*

► **Theorem 17 (Weak Completeness).** *The axiomatic system of $\mathcal{L}^+(\mathcal{A})$ is complete with respect to the Markovian semantics, i.e. if $\Vdash \psi$, then $\vdash \psi$.*

6 From bisimulation to the metric space of logical formulas

To start with, we state that the logical equivalences induced by $\mathcal{L}(\mathcal{A})$ and by $\mathcal{L}^+(\mathcal{A})$ on the class of CMPs coincide with stochastic bisimulation. The proofs follow closely the proof of the corresponding result for probabilistic systems [8, 10, 20] and consist in showing that the negation free-fragment of $\mathcal{L}(\mathcal{A})$ characterizes stochastic bisimulation while the negation and M_r^a do not differentiate bisimilar processes.

► **Theorem 18 (Logical characterization of stochastic bisimulation).** *Let $\mathcal{M} = (M, \Sigma, \tau)$, $\mathcal{M}' = (M', \Sigma', \tau') \in \mathfrak{M}$, $m \in M$ and $m' \in M'$. The following assertions are equivalent.*

1. $(\mathcal{M}, m) \sim (\mathcal{M}', m')$;
2. For any $\phi \in \mathcal{L}(\mathcal{A})$, $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{M}', m' \Vdash \phi$;
3. For any $\phi \in \mathcal{L}^+(\mathcal{A})$, $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{M}', m' \Vdash \phi$.

One of the main motivation for studying quantitative logics for probabilistic and stochastic processes was, since the first papers on this subject [17, 16], the characterization of stochastic/probabilistic bisimulation. In the context of Theorem 18, one can turn the bisimulation question into a series of model-checking problems. But the concept of stochastic/probabilistic bisimulation is a very strict concept: it only verifies whether two processes have identical behaviours. In applications we need instead to know whether two processes that may differ by a small amount in the real-valued parameters (rates or probabilities) have similar behaviours. To solve this problem a class of pseudometrics have been proposed in the literature [6, 20], to measure how similar two processes are in terms of stochastic/probabilistic behaviour.

Because these pseudometrics are quantitative extensions of bisimulation, they can be defined relying on the quantitative logics. Thus, for a class \mathfrak{P} of stochastic or probabilistic processes and for a quantitative logic \mathcal{L} that characterizes the bisimulation of processes, the pseudometric can be induced by a function $d : \mathfrak{P} \times \mathfrak{P} \rightarrow [0, 1]$ which extends the (characteristic

function of the) satisfiability relation $\Vdash: \mathfrak{P} \times \mathcal{L} \rightarrow \{0, 1\}$; the function d evaluates the "degree of satisfiability" [6, 20].

In this paper we work with the function $d: \mathfrak{P} \times \mathcal{L} \rightarrow [0, 1]$, defined below for the set \mathfrak{P} of CMPs and $\mathcal{L} = \mathcal{L}^+(\mathcal{A})$ (or $\mathcal{L} = \mathcal{L}(\mathcal{A})$).

$$\begin{aligned} d((\mathcal{M}, m), \top) &= 0, \\ d((\mathcal{M}, m), \neg\phi) &= 1 - d((\mathcal{M}, m), \phi), \\ d((\mathcal{M}, m), \phi \wedge \psi) &= \max\{d((\mathcal{M}, m), \phi), d((\mathcal{M}, m), \psi)\}, \\ d((\mathcal{M}, m), L_r^a\phi) &= \langle r, \theta(a)(m)(\llbracket\phi\rrbracket) \rangle, \\ d((\mathcal{M}, m), M_r^a\phi) &= \langle \theta(a)(m)(\llbracket\phi\rrbracket), r \rangle, \end{aligned}$$

where for arbitrary $a, b \in \mathbb{R}_+$, $\langle a, b \rangle = (a - b)/a$ if $a(a - b) > 0$ and $\langle a, b \rangle = 0$ otherwise.

The results presented in this section rely on the fact that d , as most of the functions that quantify satisfiability for stochastic or probabilistic logics, is defined on top of the transition function θ . For this reason, these results can be similarly proved for other bisimulation pseudometrics.

The first result states that d characterizes stochastic bisimulation.

► **Lemma 19.** *If $(\mathcal{M}, m), (\mathcal{M}', m') \in \mathfrak{P}$, then*

$$(\mathcal{M}, m) \sim (\mathcal{M}', m') \text{ iff } [\forall \phi \in \mathcal{L}, d((\mathcal{M}, m), \phi) = d((\mathcal{M}', m'), \phi)].$$

Proof. (\implies) Induction on ϕ . The Boolean cases are trivial and the cases $\phi = L_r^a\psi$ and $\phi = M_r^a\psi$ derive from the fact that $\theta(a)(m)(\llbracket\psi\rrbracket) = \theta'(a)(m')(\llbracket\psi\rrbracket)$.

(\impliedby) For an arbitrary $\phi \in \mathcal{L}$, $\forall r \in \mathbb{Q}$, $d((\mathcal{M}, m), L_r^a\phi) = d((\mathcal{M}', m'), L_r^a\phi)$; and for r big enough $d((\mathcal{M}, m), L_r^a\phi) = 1 - \theta(a)(m)(\llbracket\phi\rrbracket)/r$, $d((\mathcal{M}', m'), L_r^a\phi) = 1 - \theta'(a)(m')(\llbracket\phi\rrbracket)/r$. Hence, $\theta(a)(m)(\llbracket\phi\rrbracket) = \theta'(a)(m')(\llbracket\phi\rrbracket)$ which implies $(\mathcal{M}, m) \sim (\mathcal{M}', m')$. ◀

As we have anticipated, a function $d: \mathfrak{P} \times \mathcal{L} \rightarrow [0, 1]$ which characterizes bisimulation in the sense of Lemma 19, induces a distance between stochastic processes, $D: \mathfrak{P} \times \mathfrak{P} \rightarrow [0, 1]$ by

$$D(P, P') = \sup\{|d(P, \phi) - d(P', \phi)|, \phi \in \mathcal{L}\}, \text{ for arbitrary } P, P' \in \mathfrak{P}.$$

D is, indeed, a pseudometric and its kernel is the stochastic bisimulation.

► **Lemma 20.** *$D: \mathfrak{P} \times \mathfrak{P} \rightarrow [0, 1]$ defined before is a pseudometric such that*

$$D(P, P') = 0 \text{ iff } P \sim P'.$$

Similarly, one can use d to define a pseudometric $\bar{d}: \mathcal{L} \times \mathcal{L} \rightarrow [0, 1]$ over the space of logical formulas by

$$\bar{d}(\phi, \psi) = \sup\{|d(P, \phi) - d(P, \psi)|, P \in \mathfrak{P}\}, \text{ for arbitrary } \phi, \psi \in \mathcal{L}.$$

► **Lemma 21.** *$\bar{d}: \mathcal{L} \times \mathcal{L} \rightarrow [0, 1]$ defined before is a pseudometric and*

$$\bar{d}(\phi, \psi) = \bar{d}(\neg\phi, \neg\psi).$$

Proof. We prove that it satisfies the triangle inequality. We have

$$\sup\{|d((\bar{\Omega}, \Gamma), \phi) - d((\bar{\Omega}, \Gamma), \psi)|\} + \sup\{|d((\bar{\Omega}, \Gamma), \psi) - d((\bar{\Omega}, \Gamma), \rho)|\} \geq \sup\{|d((\bar{\Omega}, \Gamma), \phi) - d((\bar{\Omega}, \Gamma), \rho)|\} + \sup\{|d((\bar{\Omega}, \Gamma), \psi) - d((\bar{\Omega}, \Gamma), \rho)|\} \geq \sup\{|d((\bar{\Omega}, \Gamma), \phi) - d((\bar{\Omega}, \Gamma), \rho)|\}. \quad \blacktriangleleft$$

This construction allow us to introduce the first robustness theorem.

► **Theorem 22 (Strong Robustness).** *For arbitrary $\phi, \psi \in \mathcal{L}$ and $P \in \mathfrak{P}$,*

$$d(P, \psi) \leq d(P, \phi) + \bar{d}(\phi, \psi).$$

Proof. From the definition of \bar{d} we have that $d(P, \psi) - d(P, \phi) \leq \bar{d}(\phi, \psi)$. \blacktriangleleft

Similar constructions can be done for any class of stochastic or probabilistic models for which it has been defined a correspondent logic that characterizes bisimulation. But in spite of the obvious utility of the robustness theorem, in most of the cases such a result is not computable due to the definition of \bar{d} that involves the quantification over the entire class of continuous Markov processes.

This is exactly where the sound and complete axiomatizations of $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^+(\mathcal{A})$ for the Markovian semantics and the finite model properties play their role. In what follows, we use the construction of the small model for an \mathcal{L} -consistent formula presented in the previous section⁴ to effectively compute an approximation of \bar{d} within a given error $\varepsilon > 0$. Below we reuse the notations of section 5.

Let Ω be the set of the \mathcal{L} -maximally consistent sets of formulas. For arbitrary $\Gamma^\infty \in \Omega$, $a \in \mathcal{A}$ and $\phi \in \mathcal{L}$, let

$$\begin{aligned} a_\phi^{\Gamma^\infty} &= \sup\{r \in \mathbb{Q} : L_r^a \phi \in \Gamma^\infty\} = \inf\{r \in \mathbb{Q} : \neg L_r^a \phi \in \Gamma^\infty\} = \\ &= \inf\{r \in \mathbb{Q} : M_r^a \phi \in \Gamma^\infty\} = \sup\{r \in \mathbb{Q} : \neg M_r^a \phi \in \Gamma^\infty\}. \end{aligned}$$

The existence of these inf and sup and their equalities can be proved as in Lemma 6 (4).

► **Lemma 23** (Extended Truth Lemma). *If $\theta : \mathcal{A} \rightarrow [\Omega \rightarrow \Delta(\Omega, 2^\Omega)]$ is defined for arbitrary $a \in \mathcal{A}$, $\Gamma^\infty \in \Omega$ and $\phi \in \mathcal{L}$ by $\theta(a)(\Gamma^\infty)([\phi]) = a_\phi^{\Gamma^\infty}$, then $\mathcal{M}_\mathcal{L} = (\Omega, 2^\Omega, \theta) \in \mathfrak{M}$. Moreover, for arbitrary $\phi \in \mathcal{L}$,*

$$\mathcal{M}_\mathcal{L}, \Gamma^\infty \Vdash \phi \text{ iff } \phi \in \Gamma^\infty.$$

The proof of this lemma is the sum of the proofs of the lemmas 8, 9, 14 and 15.

The next lemma states that \bar{d} can be characterized by only using the processes of $\mathcal{M}_\mathcal{L}$. In this way it relates \bar{d} to provability, as these processes are \mathcal{L} -maximally consistent sets of formulas.

► **Lemma 24.** *For arbitrary $\phi, \psi \in \mathcal{L}$,*

$$\bar{d}(\phi, \psi) = \sup\{|d((\mathcal{M}_\mathcal{L}, \Gamma^\infty), \phi) - d((\mathcal{M}_\mathcal{L}, \Gamma^\infty), \psi)|, \Gamma^\infty \in \Omega\}.$$

Proof. Any $(\mathcal{M}, m) \in \mathfrak{M}$ satisfies a maximally-consistent set of formulas, hence there exists $\Gamma^\infty \in \Omega$ such that $(\mathcal{M}, m) \sim (\mathcal{M}_\mathcal{L}, \Gamma^\infty)$, i.e., for any $\phi \in \mathcal{L}$, $d((\mathcal{M}, m), \phi) = d((\mathcal{M}_\mathcal{L}, \Gamma^\infty), \phi)$. \blacktriangleleft

In what follows we reduce the quantification space to the domain of a finite model. For an arbitrary consistent formula $\psi \in \mathcal{L}$, let $\mathcal{M}_\psi = (\Omega_p[\psi], 2^{\Omega_p[\psi]}, \theta_\psi) \in \mathfrak{M}$ be the model of ψ constructed in the previous section; we call p the *parameter* of \mathcal{M}_ψ .

Let $\tilde{d} : \mathcal{L} \times \mathcal{L} \rightarrow [0, 1]$ be defined as follows.

$$\begin{aligned} \tilde{d}(\phi, \psi) &= \max\{|d((\mathcal{M}_{\phi \wedge \psi}, \Gamma), \phi) - d((\mathcal{M}_{\phi \wedge \psi}, \Gamma), \psi)|, \Gamma \in \Omega_p[\phi \wedge \psi]\} \text{ if } \phi \wedge \psi \text{ is consistent,} \\ \tilde{d}(\phi, \psi) &= \max\{|d((\mathcal{M}_{\neg(\phi \wedge \psi)}, \Gamma), \phi) - d((\mathcal{M}_{\neg(\phi \wedge \psi)}, \Gamma), \psi)|, \Gamma \in \Omega_p[\neg(\phi \wedge \psi)]\}, \text{ otherwise.} \end{aligned}$$

► **Lemma 25.** *For arbitrary $\phi, \psi \in \mathcal{L}$,*

$$\bar{d}(\phi, \psi) \leq \tilde{d}(\phi, \psi) + 2/p.$$

⁴ These results hold for both $\mathcal{L} = \mathcal{L}(\mathcal{A})$ and $\mathcal{L} = \mathcal{L}^+(\mathcal{A})$.

Proof. To prove the inequality, we return to the notations of lemmas 6 and 7. We have $x, y \in \mathbb{Q}_p$, $y = x + 1/p$ and $x \leq z < y$. This implies that for any $\phi \in \mathcal{L}[\psi]$, $|d((\mathcal{M}_{\mathcal{L}}, \Gamma^\infty), \phi) - d((\mathcal{M}_{\phi \wedge \psi}, \Gamma), \phi)| \leq 1/p$. Consequently, for arbitrary $\phi, \psi \in \mathcal{L}$, $|d((\mathcal{M}_{\mathcal{L}}, \Gamma^\infty), \phi) - d((\mathcal{M}_{\mathcal{L}}, \Gamma^\infty), \psi)| \leq |d((\mathcal{M}_{\phi \wedge \psi}, \Gamma), \phi) - d((\mathcal{M}_{\phi \wedge \psi}, \Gamma), \psi)| + 2/p$, which proves our inequality (if $\phi \wedge \psi$ is inconsistent we take $\mathcal{M}_{\neg(\phi \wedge \psi)}$). ◀

This last result finally allows us to prove a weaker version of the robustness theorem which evaluates $d((\mathcal{M}, m), \psi)$ from $d((\mathcal{M}, m), \phi)$, based on $\tilde{d}(\phi, \psi)$ and a given error.

► **Theorem 26 (Weak Robustness).** *For arbitrary $\phi, \psi \in \mathcal{L}$ and $P \in \mathfrak{P}$,*

$$d(P, \psi) \leq d(P, \phi) + \tilde{d}(\phi, \psi) + 2/p,$$

where p is the parameter of $\mathcal{M}_{\phi \wedge \psi}$ if $\phi \wedge \psi$ is consistent, or of $\mathcal{M}_{\neg(\phi \wedge \psi)}$ otherwise.

Because $\mathcal{M}_{\phi \wedge \psi}$ (or $\mathcal{M}_{\neg(\phi \wedge \psi)}$) is finite, $\tilde{d}(\phi, \psi)$ can be computed and the error $2/p$ can be controlled while constructing $\mathcal{M}_{\phi \wedge \psi}$. Hence, we can evaluate $d(P, \phi)$ from $d(P, \psi)$. This is useful when P is infinite or very large and it is expensive to repeatedly evaluate $d(P, \phi)$ for various ϕ . Instead, our theorem allows us to evaluate $d(P, \psi)$ from $d(P, \phi)$ that we can get, for instance, using statistical model checking techniques.

7 Conclusions and future works

We have introduced Continuous Markovian Logic, a multimodal logic designed to specify quantitative and qualitative properties of continuous Markov processes. CML is endowed with operators that approximate the rates of the labelled transitions of CMPs. This logic characterizes the stochastic bisimulation of CMPs.

We have presented two sound and complete Hilbert-style axiomatizations: one for the entire CML and one for its fragment without M_r^a -operators. These axiomatic systems are significantly different from the probabilistic case and from each other. The two completeness proofs rely on the finite model properties. The constructions of the finite models adapt the filtration method of modal logics to stochastic settings, where a series of specific problems had to be solved. The small model constructions and the complete axiomatizations allow us to approach syntactically the problems of bisimulation-distances, which before in the literature have only been treated semantically. In effect, we can define a distance between logical formulas that allows us to prove the robustness theorems.

This paper opens a series of interesting research questions regarding the relation between satisfiability, provability and metric semantics summarized in [18]. There are many open questions related to the definition of \bar{d} and to the structure of the metric space of formulas. One of the problems, that we postpone for future work, is finding a classification of the functions d to reflect properties of \bar{d} . For instance, we have a partial result showing that if d satisfies some continuity conditions, then \bar{d} characterizes the logical equivalence between positive formulas (formulas without negation), i.e., $[\bar{d}(\phi, \psi) = 0 \text{ iff } \Vdash \phi \leftrightarrow \psi]$ [18]. There exist, however, distances enjoying even stronger properties such as $[\Vdash \phi \rightarrow \psi \text{ iff } \forall P \in \mathfrak{P}, d(P, \psi) \leq d(P, \phi)]$. Each of these metrics organizes the set of logical formulas as a metric space with specific topological properties. The complete axiomatization is probably the key for classifying these structures and for understanding the relationship between the topological space of models and the topological space of logical formulas.

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